THERE ARE MANY NORMAL ULTRAFILTRES CORRESPONDING TO A SUPERCOMPACT CARDINAL*

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ABSTRACT

It is proved that if κ is supercompact, there are at least $(2^{|P_\kappa(\beta)|})^+$ normal ultrafilters over $P_\kappa(\beta)$ and if $V=H.O.D.$ exactly $2^{2^{|P_\kappa(\beta)|}}$ normal ultrafilters.

$P_\kappa(\beta)$ is the set of all non-empty subsets of β of cardinality less than κ. Its cardinality is $WP(\kappa,\beta)$. $\bar{P}$ is the order type of P as a set of ordinals.

The notion of supercompact was defined by Solovay ([3]). The basic facts about it which will appear in [3] are reproduced here for the reader's convenience (up to Lemma 8 and also Lemma 12).

DEFINITION 1. *An ultrafilter U over $P_\kappa(\beta)$ is normal (n.u.f.) if:*

a) *U is κ complete*

b) *For all $\gamma < \beta$ $\{P \mid P \in P_\kappa(\beta)\ \ \gamma \in P\} \in U$*

c) *For any function $f: P_\kappa(\beta) \to \beta$ such that $f(P) \in P$ for all P, there is $\gamma < \beta$ satisfying $\{P \mid f(P) = \gamma\} \in U$.*

DEFINITION 2. *κ is β supercompact if there is a n.u.f. over $P_\kappa(\beta)$.*

κ is supercompact if κ is β supercompact for all β.

κ is terminated in β if β is the first cardinal such that κ is not β supercompact

If U is a n.u.f., then if we take the ultrapower $[V^{P_\kappa(\beta)}/U, \varepsilon/U]$ we get a well founded model of set theory.

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In 1966–67, Solovay proved Theorem 1 for the case $\beta = \kappa$ without the condition of extendability. The same result, under a somewhat weaker assumption was proved by Namba in 1967–68. As noted by Solovay, his proof can be adapted to a general β (under weaker assumptions; if $|P_\kappa(\beta)| = \beta$ it is only needed that κ is 2^β-supercompact). Solovay's result will be published in [3].

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Let V_U be its transitive isomorph and let * be the canonical elementary embedding of V into V_U. (We shall identify $V^{P_\kappa(\beta)}/U$ and V_U where no confusion arises). The importance of being normal follows from:

LEMMA 3. *If U is a n.u.f. over $P_\kappa(\beta)$ then every subset of V_U of cardinality $\leqq \beta$ is a member of V_U.*
Note that by the method of proof of Lemma 11, we could improve Lemma 3 by substituting "every set of cardinality $\leqq WP(\kappa,\beta)$".

DEFINITON 4. *A set $A \subseteq P_\kappa(\beta)$ is called* closed *if A is closed under non-empty union of less than κ of its elements.*

A is called unbounded if for any $\alpha < \beta$ there is $P \in A$ such that $\alpha \in P$.

LEMMA 5. *Let $\kappa \leqq \beta$ and U a n.u.f. over $P_\kappa(\beta)$, then every closed unbounded set A is in U.*

PROOF. Suppose that this is not the case. Define for every $P \in P_\kappa(\beta)$

$$f(P) = \cup\{Q \mid Q \subseteq P, Q \in A\}.$$

Note that $f(P) \subseteq P$ for all P and

$$f(P) = P \text{ iff } P \in A.$$

Let $B = P_\kappa(\beta) - A$. $B \in U$ since $A \notin U$.
Define

$$g(P) = \cap\{\alpha \mid \alpha \in P - f(P)\} \quad \text{for } P \in B.$$

$$g(P) = \cap\{\alpha \mid \alpha \in P\} \quad \text{for } P \in A.$$

Clearly, $g(P) \in P$ for every $P \in P_\kappa(\beta)$, therefore there is a $\gamma < \beta$ such that $\{P \mid g(P) = \gamma\} \in U$. Then, since $B \in U$, we have $B' = \{P \mid g(P) = \gamma, P \in B\} \in U$.

Let Q be an element of A such that $\gamma \in Q$. By (b) of Definition 1 and κ completeness of U there is $Q' \in B'$ such that $Q \subseteq Q'$. But $Q \subseteq f(Q')$. So $\gamma \in f(Q')$, and therefore $g(Q') \neq \gamma$. This contradicts $Q' \in B'$.

COROLLARY 6. *If U is a n.u.f. over $P_\kappa(\beta)$ then $\kappa \in U$. (Note that $\kappa \subseteq P_\kappa(\beta)$).*

PROOF. κ is closed and unbounded in $P_\kappa(\kappa)$.

It follows that U induces a unique κ complete u.f. U' over κ which is normal. (The normality of U' follows immediately from the normality of U).

DEFINITION 7. *Let $A \subseteq P_\kappa(\beta)$ $\gamma \leqq \beta$ then $[A]_\gamma$ (the restriction of A to γ) is $\{P \cap \gamma \mid P \in A\}$. If $U \subseteq P(P_\kappa(\beta))$ then $U \mid \gamma$. (the restriction of U to γ) is $\{[A]_\gamma \mid A \in U\}$.*

LEMMA 8. *If U is a n.u.f. over $P_\kappa(\beta)$, then $U|\gamma$ is a n.u.f. over $P_\kappa(\gamma)$ and $\delta \leqq \gamma \to (U|\alpha)|\delta = U|\delta$.*

DEFINITION 9. *A n.u.f. U over $P_\kappa(\beta)$ is* extendible *if there are unbounded α's such that U is the restriction of a n.u.f. over $P_\kappa(\alpha)$. (By Lemma 8 it follows that U is extendible iff for all $\alpha \geqq \beta$ there is a n.u.f. U' over $P_\kappa(\beta)$ s.t. $U = U'|\beta$).*

DEFINITION 10. *A set a is stable of degree β if there is a function f: $P_\kappa(\beta) \to V$ s.t. for all n.u.f.'s U over $P_\kappa(\beta)$, $[f]$ represents a in $V^{P_\kappa(\beta)}/U$.*

LEMMA 11. *If a is hereditarily of cardinality $\leqq WP(\kappa, \beta)$ then a is stable of degree β.*

PROOF. By induction on the rank of a. a is of cardinality $\leqq WP(\kappa, \beta)$, let r map $P_\kappa(\beta)$ onto a. for each member of $a, r(P)$, choose by the induction hypothesis a function f_P s.t. $[f_P]$ represents $r(P)$ in $V^{P_\kappa(\beta)}/U$ for all n.u.f. U.

Define O by $O(P) = |P \cap \kappa|$. By normality, it follows easily that $[O]$ represents κ in $V^{P_\kappa(\beta)}/U$ for all *n.u.f.* U.

Define g by $g(Q) = P_{0(a)}(Q)$.

FACT. *g is "normal" in the sense that if s is a function s.t. $s(P) \in g(P)$ for all P. Then, if U is a n.u.f. over $P_\kappa(\beta)$, there is a Q* s.t. $\{P \mid s(P) = Q\} \in U$.

PROOF OF THE FACT. $s(P)$ is of cardinality $< O(P)$. Using the fact that $[O]$ represents κ, and the κ completeness of U, we get $\mu < \kappa$ s.t. $C = \{P \mid \overline{s(P)} = \mu\} \in U$. Define j_α for $\alpha < \mu$ by

$j_\alpha(P) =$ The αth member of $s(P)$ if $P \in C$

$j_\alpha(P) =$ The first member of P if $P \notin C$

for each α $j_\alpha(P) \in P$ for all P ($S(P) \subseteq P$) so by condition (c) of Definition 1 we have a γ_α s.t. $C_\alpha = \{P \mid j_\alpha(P) = \gamma_\alpha\} \in U$. Let $Q = \{\gamma_\alpha \mid \alpha < \mu\}$.

It is easily seen that $D = \{P \mid s(P) = Q\} \supseteq C \cap \bigcap_{\alpha<\mu} C_\alpha$.

Hence, $D \in U$, which proves the fact.

Now, define h by $h(P) = \{f_Q(P) \mid Q \in g(P)\}$.

Let U be n.u.f. over $P_\kappa(\beta)$. We prove that $a = [h]$ in V_U.

Let $Q \in P_\kappa(\beta)$, and $q = |Q|$, by condition (b) of Definition 1 follows that $\{P \mid Q \subseteq P; |P \cap \kappa| > q\} \in U$. But $Q \subseteq P$ $|P \cap \kappa| > q$ implies $Q \in P_{0(P)}(P)$, hence $\{P \mid Q \in g(P)\} \in U$.

Therefore, $\{P \mid f_Q(P) \in h(P)\} \supseteq \{P \mid Q \in g(P)\}$, is a member of U.

Each element of a is $r(P)$ for some P, so we have $a \subseteq [h]$. On the other hand, let t be a function s.t. $B = \{P \mid t(P) \in h(P)\} \in U$. Then define

$$s(P) = \text{The first } Q \text{ s.t. } t(P) = f_Q(P) \text{ for } P \in B,$$

$$s(P) = 0 \qquad \text{for } P \notin B.$$

(We use some fixed well-ordering of $P_\kappa(\beta)$).

For all P we have $s(P) \in g(P)$, so by 'normality' of g, s is almost constant. There is Q s.t. $\{P \mid s(P) = Q\} \in U$, so $\{P \mid s(P) = Q, \quad P \in B\} \in U$. Hence, $\{P \mid t(P) = f_Q(P)\} \in U$ which means $[t] = [f_Q]$. Thus, we proved $[h] \subseteq a$.

Denote h of Lemma 11 by f^β_a. Let $\beta \leqq \gamma$. It can be easily seen that we can choose f^γ_a and f^β_a (by choosing the r of Lemma 11 for β and γ consistently) s.t. $f^\gamma_a(P) = f^\beta_a(P \cap \beta)$.

Note that $f^\beta_\alpha(P) < \kappa$ where $\alpha \leqq \beta$.

LEMMA 12. *Let U be a n.u.f. over $P_\kappa(\beta)$, $\gamma \leqq \beta$ then $V_{U|\gamma}$ can be elementarily embedded in V_U in such a way that sets, hereditarily of cardinality $\leqq WP(\kappa,\gamma)$ are preserved.*

PROOF. We embed $V^{P_\kappa(\beta)}/U \mid \gamma$ in $V^{P_\kappa(\beta)}/U$ by $i([f]) = [f']$ where f' is given by $f'(P) = f(P \cap \gamma)$, for all $P \in P_\kappa(\beta)$. i is well defined because if

$$A = \{P \mid f(P) = f_1(P)\} \in U \mid \gamma$$

then there is $B \in U$ such that $A = [B]_\gamma$ but $B \subseteq \{P \mid f'(P) = f'_1(P)\} \rightarrow [f'] = [f'_1]$.

i is elementary embedding because
$V^{P_\kappa(\gamma)}/U \mid \gamma \models l([f_1] \cdots [f_n])$ iff
$A = \{P \mid V \models l(f_1(P), \cdots f_n(P))\} \in U \mid \gamma$.

$A \in U \mid \gamma$ implies that there is $B \in U$ such that $A = [B]_\gamma$. So we have

$$B \subseteq \{P \mid V \models l(f_1(P), \cdots f_n(P)\} = C,$$

then $C \in U$, that is to say

$$V^{P_\kappa(\beta)}/U \models l([f'_1], \cdots [f'_n]).$$

The note after Lemma 11 actually proves that all sets, hereditarily of cardinality $\leqq WP(\kappa,\gamma)$ are preserved by i.

DEFINITION 13. *α is* good *if α is a strong limit cardinal ($\beta < \alpha \rightarrow 2^\beta < \alpha$) and if $cf(\alpha) \geqq \kappa$. If α is good then $|P_\kappa(\beta)| = \alpha$.*

Let κ^*_U be the image of κ under $*$ in V_U. From the note in the proof of Lemma 11, it follows that $\beta < \kappa^*$.

LEMMA 14. *Let U, U' be n.u.f. over $P_\kappa(\beta)$. Suppose that* $|P_\kappa(\beta)| = \beta$. *If $U' \in V_U$ then $\kappa_{U'}^* < \kappa_U^*$.*

PROOF. Since κ is inaccessible, $V_U \models \kappa_U^*$ is inaccessible. All the functions $f: P_\kappa(\beta) \to \kappa$ are in V_U (they are hereditarily of cardinality β). But in V_U, $V_U \models \kappa_U^* > \beta \geqq \kappa$, therefore in V_U the set $\kappa^{P_\kappa(\beta)}$ has cardinality less than κ_U^*.

Since $\kappa_U^{*\prime} = \{[f] \mid f \in \kappa^{P_\kappa(\kappa)}\}$, and $U' \in V_U$ the map $f \to [f]$ is in V_U, so in $V_{U'}$ $|\kappa_{U'}^*| < |\kappa_U^*| \Rightarrow \kappa_{U'}^* < \kappa_U^*$.

LEMMA 15. *If α is good and κ is α supercompact then there is a n.u.f. ,U, over $P_\kappa(\alpha)$ such that in V_U κ is terminated in α.*

PROOF. *Since α is a strong limit cardinal, for any n.u.f. U over $P_\kappa(\alpha)$, we have $V_U \models \kappa$ is β supercompact for all $\beta < \alpha$. (A n.u.f. over $P_\kappa(\beta)$ is a set hereditarily of cardinality $< \alpha$).*

Choose U over $P_\kappa(\alpha)$ *s.t.* κ_U^* is minimal, $|P_\kappa(\alpha)| = \alpha$, therefore $P(P_\kappa(\alpha)) \in V_U$. (For any subset of $P_\kappa(\alpha)$ is hereditarily of cardinality $\leqq \alpha$).

If we have $V_U \models \kappa$ is α supercompact, then $V_U \models$ there is a n.u.f. ofer $P_\kappa(\alpha)$, but if U' is a n.u.f. over $P_\kappa(\alpha)$ in V_U, it is a *n.u.f.* in V. ($P(P_\kappa(\alpha) \in V_U$!)

Since $U' \in V_U$, by Lemma 14 we get $\kappa_{U'}^* < \kappa_U^*$ which contradicts the minimality of κ_U^*.

DEFINITION 16. *Let κ be terminated in β. α* reflects *κ if the following holds: For all $\gamma < \alpha$. If U is a n.u.f. over $P_\kappa(\gamma)$ and for all $\gamma \leqq \delta < \alpha$, U is a restriction of a n.u.f. over $P_\kappa(\delta)$, then for all $\varepsilon < \beta$, U is the restriction of a n.u.f. over $P_\kappa(\varepsilon)$. If κ is supercompact, then α* reflects *κ if for all $\gamma < \alpha$, if U is n.u.f. over $P_\kappa(\gamma)$ and for all $\gamma \leqq \delta < \alpha$ U is a restriction of a n.u.f., over $P_\kappa(\delta)$, then U is extendible. The usual reflection principle* ([1]) *shows that if κ is supercompact, we have unbounded numbers of α's which are good and reflect κ.*

Suppose that α is good and reflects κ, U a n.u.f. over $P_\kappa(\alpha)$ such that $V_U \models \kappa$ is terminated in α. Then for $\gamma < \alpha$:

$$V \models \gamma \text{ is good and reflects } \kappa \text{ iff } V_U \models \gamma \text{ is good and reflects } \kappa.$$

The proof is straightforward by noting that all subsets hereditarily of cardinality $\leqq \alpha$ are in V_U.

Under the same assumptions if α is the γth good cardinal which reflects κ we get by the same consideration (note $\gamma \leqq \alpha$).

$V_U \models \kappa$ is finished in the γth cardinal which is good and reflects κ.

THEOREM 1. *If κ is supercompact, there are al least $(2^{WP(\kappa,\beta)})^+$ n.u.f.'s over $P_\kappa(\beta)$ which are extendible.*

PROOF. We shall map $(2^{WP(\kappa,\beta)})^+$ into the set of extendible n.u.f.'s in a one-to-one manner.

Let $\gamma < (2^{WP(\kappa,\beta)})^+$ and let α_γ be the γth good cardinal $\geqq \beta$ which reflects κ. Let U_γ be a n.u.f. s.t. in $V_U \models \kappa$ is terminated in α_γ. (U_γ exists by Lemma 15).

$V_{U_\gamma} \models \kappa$ is terminated in γth cardinal which is good, $\geqq \beta$, and reflects κ. Hence:

$V_U \models$ there is a well-ordering X of $P(P_\kappa(\beta))$ *s.t.* if γ_X is the order type of X, κ is terminated in α_{γ_X}.

By Lemma 12 we have:

(I). $V_{U|\beta} \models$ There is a well-ordering X of $P(P_\kappa(\beta))$ *s.t.* γ_X is the order type of X, κ is terminated in α_{γ_X}. (Note that β and κ are preserved by the elementary embedding of $V_{U|\beta}$ into V_U).

Let $<(P_\kappa(\beta)),\boldsymbol{\alpha}>$ be that well-ordering in $V_{U|\beta}$.

It must be of order type γ because the elementary embedding of $V_{U|\beta}$ into V_{U_γ} preserves $\boldsymbol{\alpha}$. (It preserves any element of $P(P_\kappa(\beta))$, which means that it preserves pairs of elements of $P(P_\kappa(\beta))$, and any set of pairs of elements of $P(P_\kappa(\beta))$).

Clearly the mapping $\gamma \to U_\gamma | \beta$, is one-to-one because for different α's we get different well-ordering fulfilling the statement I.

$U_\gamma | \beta$ is extendible because it is a restriction of a n.u.f. over $P_\kappa(\alpha_\gamma)$, and α_γ reflects κ, which means that $U_\gamma | \beta$ is extendible to any ordinal $< \alpha_\gamma \Rightarrow U_\gamma | \beta$ is extendible. Q.E.D.

In Theorem 2 we give an exact figure for the cardinality of n.u.f.'*s* over $P_\kappa(\beta)$. However, we use an assumption $V = H.O.D.$ where $V = H.O.D.$ is a shorthand for "every set is ordinal definable", ([2]), which is by no means known to be consistent with the existance of supercompact cardinal.

THEOREM 2. *If $V = H.O.D.$ and κ is supercompact, then there are $2^{2^{WP(\kappa,\beta)}}$ extendible n.u.f.'s over $P_\kappa(\beta)$.*

PROOF. Define α to be *super-good* if $\alpha = |R(\alpha)|$, every set in $R(\alpha)$ is ordinal definable in $R(\alpha)$ for some $\beta < \alpha$, and $cf(\alpha) \geqq \kappa$.

If in $R(\beta)$ every set is ordinal definable in some $R(\beta)$ ($\beta < \alpha$) then the canonical well-ordering of $R(\beta)$ described in ([2]) is the restriction of the canonical well-ordering of V.

If α is super-good, U a *n.u.f.* over $P_\kappa(\alpha)$ then $R(\alpha) \subseteq V_U$ (because $R(\alpha)$ is hereditarily of cardinality $\leqq \alpha$), and $V_U \models \alpha$ is super-good.

We map $2^{2^{WP(\kappa,\beta)}}$ to the set of n.u.f.'s over $P_\kappa(\beta)$ which are extendible.

Let $\gamma < 2^{2^{WP(\ \ \beta)}}$ and α_γ the γ-th super-good cardinal which reflects κ and is $\geqq \beta$.

Let U_γ be a n.u.f. over $P_\kappa(\alpha_\gamma)$ s.t. $V_U \models \kappa$ is terminated in α_γ. Since $(\gamma < 2^{2^{WP(\cdots\beta)}}$ we have a canonical well-ordering of $P(P(P_\kappa(\beta)))$ of order types $2^{2^{WP(\cdots\beta)}}$. This well-ordering is definable in $R(\alpha_\gamma)$, so it is definable in V_{U_γ}. Denote it by $\boldsymbol{\alpha}$. Then γ is the order type of an initial segment of $\boldsymbol{\alpha}$ fixed by $a \in P(P(P_\kappa(\beta)))$.

$a \in V_{U\,|\beta}$ because a is definable from β and κ in V_U and therefore it is the image of some element of $V_{U\gamma|\beta}$ by i of Lemma 12.

But $a \subseteq P(P_\kappa(\beta))$ and each of the elements of $P(P_\kappa(\beta))$ is preserved by i, so $i^{-1}(a) = a$.

For different γ's we get different a's because an initial segment of order type $\boldsymbol{\alpha}$ is fixed in the canonical well-ordering of $P(P(P_\kappa(\beta)))$ by a different subset of $P(P_\kappa(\beta)$.)

Each of the a's is definable in $V_{U\,|\beta}$ from κ and β by the same formula, which implies by the stability of κ and β of degree β that:

$$\gamma \neq \gamma' \Rightarrow U_\gamma \,|\, \beta \neq U_{\gamma'} \,|\, \beta.$$

$U_\gamma \,|\, \beta$ is extendible because it is extendible up to α_γ and α_γ reflects κ. Q.E.D.

By Corollary 6, we get that κ as a measurable cardinal has at least $(2^\kappa)^+$ normal ultrafilters and if $V = H.O.D.$, exactly 2^{2^κ} n.u.f.'s.

References

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